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Rigorous real-time Feynman path integral for vector potentials

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Abstract. In this paper, we will show the existence and uniqueness of a real-time, time-sliced Feynman path integral for quantum systems with vector potential. Our formulation of the path integral will be derived on the L^2 transition probability amplitude via improper Riemann integrals. Our formulation will hold for a vector potential Hamiltonian for which its potential and vector potential each carry at most a finite number of singularities and discontinuities.

1. Introduction

In his paper (see [20, footnote 13]), Feynman observed that by using wavefunctions, ill-defined oscillatory integrals can be given a rigorous meaning. We will use Feynman's observation and use wavefunctions to provide a convergence factor in the derivation of a real-time propagator that takes the form of an L^2 transition probability amplitude and the derivation of a real-time, time-sliced Feynman path integral.

Since Feynman's invention of the path integral in the 1940s (see [10, 20]), giving the real-time Feynman path integral a rigorous mathematical justification for general potentials has been a stumbling block (see [1–3, 6–8, 19], and references within). In physics, the real-time, time-sliced Feynman path integral is formulated on the propagator with improper Riemann integrals in the hope of convergence since the integrand of the path integral in real time is not absolutely integrable in a Lebesgue sense (see [13, 18] and references therein). In the spirit of physics, in a previous work [15], we formulated a rigorous real-time, time-sliced Feynman path integral on the L^2 transition probability amplitude via improper Riemann integrals. Our previous formulation held for non-vector potential Hamiltonians with a potential that has at most a finite number of singularities and discontinuities.

In this paper, we will extend our previous work to vector potential Hamiltonians with a potential and vector potential that carry at most a finite number of singularities and discontinuities.

In physics, the vector potential Feynman path integral for the propagator is formally given by

$$\lim_{k \rightarrow \infty} \left(\frac{m}{2i\pi\hbar\epsilon} \right)^{n(k+1)/2} \int_{\mathbb{R}^{nk}} \exp \left\{ \frac{i\epsilon}{\hbar} \sum_{j=0}^k \left[\frac{m}{2} \left(\frac{\vec{x}_{j+1} - \vec{x}_j}{\epsilon} \right)^2 - V(\vec{x}_j) \right] + \frac{ie}{\hbar c} \sum_{j=0}^k (\vec{x}_{j+1} - \vec{x}_j) * \vec{a} \left(\frac{\vec{x}_{j+1} + \vec{x}_j}{2} \right) \right\} d\vec{x}_1 \dots d\vec{x}_k \quad (1.1)$$

where the integrals in (1.1) are improper Riemann integrals (see [13, 18] and references therein) and the $*$ inside the integral is the vector dot product. In this paper, we will derive the following:

$$\begin{aligned}
 \langle \phi^*, \exp(-it\bar{H}) \psi \rangle_{L^2} &= \int_{\mathbb{R}^n} \phi(\vec{x}) [\exp(-it\bar{H}) \psi](\vec{x}) \, d\vec{x} \\
 &= \lim_{k \rightarrow \infty} \left\{ \left(\frac{1}{4i\pi\epsilon} \right)^{[n(k-1)]/2} \int_{r\mathbb{R}^{n(k+1)}} \phi(\vec{x}_k) \exp \left\{ i\epsilon \sum_{j=0}^{k-1} \left[\frac{1}{4} \left(\frac{\vec{x}_{j+1} - \vec{x}_j}{\epsilon} \right)^2 \right. \right. \right. \\
 &\quad \left. \left. \left. - V(\vec{x}_{j+1}) + \frac{\bar{\lambda}(\vec{x}_{j+1}, \vec{x}_j)}{\epsilon} \right] \right\} \psi(\vec{x}_0) \, d\vec{x}_0 \dots d\vec{x}_k \right\}. \tag{1.2}
 \end{aligned}$$

In (1.2), we have used the notation $r\mathbb{R}^n$ in the second line to mean an improper Riemann integral over \mathbb{R}^n ; $\epsilon = \frac{t}{k}$; \bar{H} is the closure of the essentially self-adjoint Hamiltonian

$$H = \sum_{j=1}^n (-i\nabla_j - \vec{a}_j)^2 + V \tag{1.3}$$

$\phi, \psi \in L^2$; ϕ, ψ, \vec{a}, V are such that each has at most a finite number of discontinuities and singularities, and $\bar{\lambda}$ is dependent on \vec{a} .

We point out that the function $\bar{\lambda}$ does not reproduce the term $(\vec{x}_{j+1} - \vec{x}_j) * \vec{a}((\vec{x}_{j+1} + \vec{x}_j)/2)$ in (1.1), but we will see that when the vector potential is well behaved and when \vec{x}_{j+1}, \vec{x}_j are close to each other, $\bar{\lambda}(\vec{x}_{j+1}, \vec{x}_j)$ is close to $(\vec{x}_{j+1} - \vec{x}_j) * \vec{a}((\vec{x}_{j+1} + \vec{x}_j)/2)$. Lastly, both the vector potential and scalar potential can carry any kind of singularity as long as the Hamiltonian in (1.3) is essentially self-adjoint. One possible and interesting application could be the Coulomb potential problem. Duru and Kleinert (see [13] and references within) formally computed the hydrogen atom propagator by using integration over a path space version of the Feynman path integral, they produced the correct energy eigenvalues. Kleinert [13] used the time-sliced version of the Feynman path integral to formally justify the integration over path space calculation but mathematical rigour is still lacking.

The idea for the derivation of (1.2) is the following. For suitable operators A and B , Trotter's product formula reads (see [5, 11, 16, 17])

$$\exp(-itC) = s - \lim_{k \rightarrow \infty} \left(\exp\left(\frac{-itA}{k}\right) \exp\left(\frac{-itB}{k}\right) \right)^k \tag{1.4}$$

where C is the closure of $A + B$. Together with the work of Simon (see [19] and lemma 2.4 below), it would be reasonable to believe that for the Hamiltonian in (1.3),

$$\exp(-it\bar{H}) = s - \lim_{k \rightarrow \infty} \left(\exp\left(\frac{-itV}{k}\right) \prod_{j=1}^n e^{i\lambda_j} \exp\left(\frac{-itH_0}{k}\right) e^{-i\lambda_j} \right). \tag{1.5}$$

We will use a generalized Trotter's formula due to Chernoff (see [8]) to prove (1.5) for (essentially self-adjoint) H that satisfies theorem 2.2 or theorem 2.3 below. From here on, we will assume that H is such that theorem 2.2 or theorem 2.3 below holds.

We will derive (1.2) with the help of (1.5) and the following idea. For simplicity, suppose $f(x) \in L^2(\mathbb{R}), g_t(x, y) \in L^2(\mathbb{R} \times \mathbb{R})$ are such that they are bounded and continuous. Furthermore, suppose that both

$$\begin{aligned}
 h_t(x) &= \int_{-a}^b g_t(x, y) \, dy \\
 p_t(x) &= s - \lim_{a, b \rightarrow \infty} \int_{-a}^b g_t(x, y) \, dy
 \end{aligned} \tag{1.6}$$

are in $L^2(\mathbb{R})$ as a function of x . In (1.6), we take the integral to be Lebesgue integrals and the limits are taken independently of each other. Note that for $p_t(x)$, we can interpret the integral as an improper Lebesgue integral with convergence in the L^2 topology. Let us denote $\chi_{[-c,d]}$ to be the characteristic function on $[-c, d]$. Schwarz's inequality then implies

$$\left| \int_{\mathbb{R}} f(x)p_t(x) dx - \int_{-c}^d \int_{-a}^b f(x)g_t(x, y) dx dy \right| \leq \|f\|_2 \|p_t - h_t\|_2 + \|f - \chi_{[-c,d]}f\|_2 \|h_t\|_2 \rightarrow 0. \tag{1.7}$$

Thus, we can write

$$\int_{\mathbb{R}} f(x)p_t(x) = \lim_{a,b,c,d \rightarrow \infty} \int_{-c}^d \int_{-a}^b f(x)g_t(x, y) dx dy \tag{1.8}$$

where the limits are all taken independently of each other. Since f and g are bounded and continuous, the Lebesgue integral over $[-a, b] \times [-c, d]$ in (1.8) can be replaced by a Riemann integral. Since the limits are taken independently of each other, we can then interpret the right-hand side of (1.8) as an improper Riemann integral. If f and g carry singularities and discontinuities, care must be taken in the region of integration so that the replacement of the Lebesgue integral with Riemann integrals can be done. What we have demonstrated in our simple example is just a way to turn convergence in L^2 topology into pointwise convergence in t by integrating against another L^2 function. In mathematics, there is a rigorous real-time, time-sliced path integral where the convergence of the improper Lebesgue integrals are taken in the L^2 topology. We will use the above idea to convert the L^2 convergence path integral into improper Riemann integrals.

2. Background

In this section, we will provide some background with references; we leave it to the reader to look up the proofs. The following is a generalized Trotter product formula due to Chernoff [8]. We will use a modified version of the theorem to prove (1.5).

Theorem 2.1. *Let $F(t), t \geq 0$ be a family of linear contractions in X with $F(0) = I$. If the closure C of the strong derivative*

$$F'(0) = s - \lim_{\epsilon \rightarrow 0} \frac{F(\epsilon) - I}{\epsilon} \tag{2.1}$$

generates a contractive semigroup, then $F(t/k)^k \rightarrow e^{tC}$ strongly and uniformly on bounded t intervals.

Proof. See [8]. □

We are interested in the essential self-adjointness of (1.3); the following two theorems provide conditions on \vec{a} and V for essential self-adjointness. For more details on the subject, see [4, 9, 12, 14, 19] and references therein.

Theorem 2.2. *Let $n \geq 4$ and $p = 6n/(n + 2)$. Let \vec{a} be an n -dimensional vector in \mathbb{R}^n and each component of \vec{a} is in L^p_{loc} . Furthermore, let $V, \nabla \vec{a} \in L^{p/2}_{loc}$, then $(-i\nabla - \vec{a})^2 + V$ is essentially self-adjoint on $C^\infty_0(\mathbb{R}^n)$*

Proof. See [19]. □

Theorem 2.3. *Suppose that each component of \vec{a} is a real-valued function in $L^4(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $\nabla * \vec{a} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, and V is a real-valued function in $L^2 + L^\infty$. For $\phi \in C_0^\infty(\mathbb{R}^3)$, define*

$$H\phi = -\Delta\phi + 2i\vec{a} * \nabla\phi + i(\nabla * \vec{a})\phi + V\phi + \vec{a}^2\phi \tag{2.2}$$

then H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3)$.

Proof. See [17]. □

The following lemma is due to Simon [19]. We will use the idea to show (1.5).

Lemma 2.4. *Let $a_j \in L^2_{loc}(\mathbb{R}^n)$. Then $-i\partial_j - a_j$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and its closure $-iD_j$ obeys*

$$-iD_j = e^{i\lambda_j}(-i\partial_j)e^{-i\lambda_j} \tag{2.3}$$

for a real-valued function $\lambda_j \in L^2_{loc}(\mathbb{R}^n)$. Furthermore, the domain of D_j is

$$\mathcal{D}_j(D_j) = \{\phi \in L^2 \mid (\partial_j - ia_j)\phi \text{ (dist. sense)} \in L^2\} \tag{2.4}$$

and

$$\lambda_j(x_1, \dots, x_n) = \int_0^{x_j} a_j(x_1, \dots, x_{j-1}, y, \dots, x_n) dy. \tag{2.5}$$

Proof. See [19]. □

Since $L^p_{loc}(\mathbb{R}^n) \subset L^q_{loc}(\mathbb{R}^n)$ for $p \geq q$, any \vec{a} that satisfies theorem 2.2 or theorem 2.3 will satisfy lemma 2.4. For any \vec{a} that satisfies theorem 2.2 or theorem 2.3, lemma 2.4 implies that for $\phi \in C_0^\infty(\mathbb{R}^n)$, the following holds:

$$\begin{aligned} [e^{i\lambda_j}(-i\partial_j)e^{-i\lambda_j}]^2\phi &= e^{i\lambda_j}(-\partial_j^2)e^{-i\lambda_j}\phi = (-i\nabla_j - a_j)^2\phi \\ &= -\Delta_j\phi + 2ia_j\nabla_j\phi + i(\nabla_j a_j)\phi + a_j^2\phi \in L^2(\mathbb{R}^n). \end{aligned} \tag{2.6}$$

Let us denote $H_0^j\phi = -\Delta_j\phi$ for $\phi \in C_0^\infty(\mathbb{R}^n)$, then we can write

$$-ie^{i\lambda_j}(-\partial_j^2)e^{-i\lambda_j}\phi = \lim_{\epsilon \rightarrow 0} \left\{ e^{i\lambda_j} \frac{\exp(-i\epsilon H_0^j) - I}{\epsilon} e^{-i\lambda_j}\phi \right\}. \tag{2.7}$$

Proposition 2.5. *Let \vec{a} and V be as in theorem 2.2 or theorem 2.3. Denote*

$$F_j(t) = e^{-itV} \prod_{k=1}^j e^{i\lambda_k} \exp(-itH_0^k) e^{-i\lambda_k} \tag{2.8}$$

then for $\phi \in C_0^\infty(\mathbb{R}^n)$,

$$F_j'(0)\phi = -i \left(\sum_{k=1}^j (-i\nabla_k - a_k)^2 + V \right) \phi \quad 1 \leq j \leq n. \tag{2.9}$$

Proof. Equation (2.6) and (2.7) implies that (2.9) is true for $j = 1$. We will show (2.9) by induction up to $j = n$. Let us denote

$$\begin{aligned} \bar{F}(\epsilon) &= e^{-i\epsilon V} \exp(i\lambda_{j+1}) \exp(-i\epsilon H_0^{j+1}) \exp(-i\lambda_{j+1}) \\ \bar{F}_j(\epsilon) &= \prod_{k=1}^j e^{i\lambda_k} \exp(-i\epsilon H_0^k) e^{-i\lambda_k}. \end{aligned} \tag{2.10}$$

Suppose (2.9) is true for $1 \leq j < n$, then

$$\begin{aligned} F'_{j+1}(0)\phi &= \lim_{\epsilon \rightarrow 0} \frac{(\bar{F}(\epsilon)\bar{F}_j(\epsilon) - I)\phi}{\epsilon} \\ &= \lim_{\epsilon \rightarrow 0} \frac{(\bar{F}(\epsilon)\bar{F}_j(\epsilon) - \bar{F}(\epsilon))\phi}{\epsilon} + \lim_{\epsilon \rightarrow 0} \frac{(\bar{F}(\epsilon) - I)\phi}{\epsilon} \\ &= -i \left(\sum_{k=1}^{j+1} (-i\nabla_k - a_k)^2 + V \right) \phi. \end{aligned} \tag{2.11}$$

□

We cannot immediately apply theorem 2.1 on this $F_n(t)$ to produce a Trotter product formula for (1.5) since we only know the behaviour of $F'(0)$ on C_0^∞ , while theorem 2.1 requires knowledge of the behaviour of $F'(0)$ on its domain. We will prove a slightly different version of theorem 2.1 which only requires knowledge of $F'(0)$ on a dense subset of its domain. The proof is a small modification of Chernoff's proof of theorem 2.1

3. Generalized Trotter product formula

We quote a few results from Chernoff [8]; we leave it to the reader to look up the proofs.

Lemma 3.1. Let $C_n, n = 1, 2, \dots$, and C be the generators of (C_0) contraction semigroups on X . Let \mathcal{D} be a dense subspace of the domain of C such that $C\mathcal{D} = C$. Suppose that for all $\phi \in \mathcal{D}$, $C_n\phi$ is defined and $\lim_{n \rightarrow \infty} C_n\phi = C\phi$. Then for every $\lambda > 0$, $(\lambda - C_n)^{-1}$ converges to $(\lambda - C)^{-1}$ in the strong operator topology.

Proof. See [8]. □

Proposition 3.2. Under the hypothesis of lemma 3.1, $e^{tC_n} \rightarrow e^{tC}$ in the strong operator topology, the convergence being uniform on every compact interval.

Proof. See [8]. □

Lemma 3.3. Let T be a linear contraction on X . Then $t \rightarrow e^{t(T-I)}$ is a contraction semigroup. For all $\phi \in X$ we have $\|(e^{t(T-I)} - T^k)\phi\| \leq k^{1/2}\|(T - I)\phi\|$.

Proof. See [8]. □

The following is the result that we seek.

Theorem 3.4. Let $F(t)$ be a strongly continuous function from $[0, \infty)$ to the linear contractions on X such that $F(0) = I$. Suppose that $F'(0)$ is defined on a dense subset \mathcal{D} of X and its closure C generates a contractive semigroup. Then $F^k(t/k)$ converges to e^{tC} in the strong operator topology.

Proof. Fix $t > 0$. Define $C_k = k(F(t/k) - I)/t$. Lemma 3.3 implies that

$$e^{k(F(t/k)-I)} = e^{tC_k} \tag{3.1}$$

exists. The hypotheses of lemma 3.1 are satisfied, hence proposition 3.2 implies that for all $\phi \in X$,

$$e^{tC_k}\phi \rightarrow e^{tC}\phi. \tag{3.2}$$

Suppose that $\phi \in \mathcal{D}$. Lemma 3.3 implies that

$$\begin{aligned} \left\| e^{tC_k}\phi - F^k\left(\frac{t}{k}\right)\phi \right\| &\leq k^{1/2} \left\| \left(F\left(\frac{t}{k}\right) - I\right)\phi \right\| \\ &= \frac{t}{k^{1/2}} \left\| \frac{k}{t} \left(F\left(\frac{t}{k}\right) - I\right)\phi \right\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \tag{3.3}$$

Hence for $\phi \in \mathcal{D}$, $F^k(t/k)\phi \rightarrow e^{tC}\phi$. Since the operators are contractions, we have the result of all $\phi \in X$. \square

The proof of theorem 3.4 is exactly that of theorem 3.1 except that Chernoff let \mathcal{D} be the domain of $F'(0)$ where as we took \mathcal{D} to be any dense subset of X for which $F'(0)$ is well defined. We can apply theorem 3.4 to the operator F_n as defined in proposition 2.5.

Theorem 3.5. Let \vec{a} and V satisfy the conditions of either theorem 2.2 or theorem 2.3. Let $F(t) = F_n(t)$, where $F_n(t)$ is as defined in proposition 2.5 and denote \bar{H} to be the closure of H , then for all $\phi \in L^2(\mathbb{R}^n)$,

$$\lim_{k \rightarrow \infty} F^k\left(\frac{t}{k}\right)\phi = e^{-it\bar{H}}\phi. \tag{3.4}$$

Proof. Proposition 2.5 implies that $F(t)$ satisfies the conditions in theorem 3.4 with $F'(0) = -iH$ and $\mathcal{D} = C_0^\infty(\mathbb{R}^n)$. The closure of $F'(0) = -iH$ is $-i\bar{H}$ and it generates a contractive semigroup $e^{-it\bar{H}}$. Hence (3.4) follows. \square

4. Rigorous real-time Feynman path integral for evolution

We are now ready to derive Feynman path integrals for vector potential Hamiltonians. We will assume that \vec{a} and V have at most a finite number of discontinuities and singularities. Suppose we take two L^2 functions ϕ and ψ such that they have at most a finite number of singularities and discontinuities and that $\{\vec{w}_1, \dots, \vec{w}_p\}$ are all the singular and discontinuous points of \vec{a}, V, ϕ and ψ . For $1 \leq \alpha \leq p$ and $1 \leq \beta \leq n$, let w_α^β be the β th coordinate of \vec{w}_α . For any fixed $k \in \mathbb{N} \cup \{0\}$, let $a_\beta^k, \bar{a}_\beta^k, b_\beta^{\alpha,k}$ and $d_\beta^{\alpha,k} \in \mathbb{R}^+$. Let

$$C_{\{j_\beta^k\}} = (-a_\beta^k, \bar{a}_\beta^k) - \bigcup_{\alpha=1}^p \left(w_\alpha^\beta - \frac{1}{b_\beta^{\alpha,k}}, w_\alpha^\beta + \frac{1}{d_\beta^{\alpha,k}} \right). \tag{4.1}$$

We will use the notation,

$$\{j_\beta^k\} = \left\{ a_\beta^k, \bar{a}_\beta^k, b_\beta^{1,k}, d_\beta^{1,k}, \dots, b_\beta^{p,k}, d_\beta^{p,k} \right\} \tag{4.2}$$

and for any set of numbers $\{J\}$, we will denote by $\{J\} \rightarrow \infty$ that each element of the set goes to infinity independently of each other. Note that if we let $\{j_\beta^k\} \rightarrow \infty$, we obtain $\mathbb{R} - \{\vec{w}_1, \dots, \vec{w}_p\}$ in (4.1). Furthermore, let

$$C_{\{j_k\}} = C_{\{j_1^k\}} \times \dots \times C_{\{j_n^k\}}$$

$$\{j_k\} = \bigcup_{\beta=1}^n \{j_\beta^k\} \tag{4.3}$$

and

$$D_k = C_{\{j_0\}} \times \dots \times C_{\{j_k\}}$$

$$\{J_k\} = \bigcup_{l=0}^k \{j_l\}. \tag{4.4}$$

We will denote the characteristic function of $C_{\{j_l^k\}}$, $C_{\{j_k\}}$, and D_k by $\chi_{C_{\{j_l^k\}}}$, $\chi_{C_{\{j_k\}}}$ and χ_{D_k} , respectively. With the above notation, for $\psi \in L^2$, we can write

$$F\left(\frac{t}{k}\right)\psi = \exp\left\{\frac{-itV}{k}\right\} \prod_{l=1}^n e^{i\lambda_l} \exp\left(\frac{-itH_0^l}{k}\right) e^{-i\lambda_l} \psi$$

$$= \exp\left\{\frac{-itV}{k}\right\} \prod_{l=1}^n e^{i\lambda_l} \exp\left(\frac{-itH_0^l}{k}\right) \lim_{\{j_l^0\} \rightarrow \infty} \chi_{C_{\{j_l^0\}}} e^{-i\lambda_l} \psi$$

$$= \lim_{\{j_0\} \rightarrow \infty} \exp\left\{\frac{-itV}{k}\right\} \prod_{l=1}^n e^{i\lambda_l} \exp\left(\frac{-itH_0^l}{k}\right) \chi_{C_{\{j_l^0\}}} e^{-i\lambda_l} \psi. \tag{4.5}$$

In (4.5), all limits are taken independently of each other and the second equality is due to the fact that all operators are continuous from L^2 to L^2 . For $1 \leq l \leq n$, $\epsilon = t/k$ and $\phi \in L^2$, we can write

$$e^{i\lambda_l} \exp(-i\epsilon H_0^l) \chi_{C_{\{j_l^0\}}} e^{-i\lambda_l} \phi = \left(\frac{1}{4i\pi\epsilon}\right)^{1/2} \exp\left[i \int_0^{x_1^l} a_l(x_0^1, \dots, x_0^{l-1}, y, \dots, x_0^n) dy\right]$$

$$\times \int_{C_{\{j_l^0\}}} \left\{ \exp\left[\frac{i\epsilon}{4} \left(\frac{x_1^l - x_0^l}{\epsilon}\right)^2\right] \right.$$

$$\left. \times \exp\left[-i \int_0^{x_0^l} a_l(x_0^1, \dots, x_0^{l-1}, y, \dots, x_0^n) dy\right] \phi(\vec{x}_0) \right\} dx_0^l$$

$$= \left(\frac{1}{4i\pi\epsilon}\right)^{1/2} \int_{C_{\{j_l^0\}}} \exp\left[\frac{i\epsilon}{4} \left(\frac{x_1^l - x_0^l}{\epsilon}\right)^2\right] \exp[i\bar{\lambda}_l(\vec{x}_1, \vec{x}_0)] \phi(\vec{x}_0) dx_0^l \tag{4.6}$$

where x_0^j and x_1^j are the j th coordinates of the vectors \vec{x}_0 and \vec{x}_1 , respectively, and

$$\bar{\lambda}_l(\vec{x}_1, \vec{x}_0) = \int_0^{x_1^l} a_l(x_0^1, \dots, x_0^{l-1}, y, \dots, x_0^n) dy - \int_0^{x_0^l} a_l(x_0^1, \dots, x_0^{l-1}, y, \dots, x_0^n) dy$$

$$= \int_{x_0^l}^{x_1^l} a_l(x_0^1, \dots, x_0^{l-1}, y, \dots, x_0^n) dy. \tag{4.7}$$

Equations (4.5) and (4.6) imply that

$$\begin{aligned}
 F\left(\frac{t}{k}\right)\psi &= \lim_{\{j_0\} \rightarrow \infty} \left\{ \exp\left\{\frac{-itV(\vec{x}_1)}{k}\right\} \left(\frac{1}{4i\pi\epsilon}\right)^{n/2} \right. \\
 &\quad \times \int_{C_{\{j_0\}}} \exp\left[\frac{i\epsilon}{4}\left(\frac{\vec{x}_1 - \vec{x}_0}{\epsilon}\right)^2 + i\sum_{l=1}^n \bar{\lambda}_l(\vec{x}_1, \vec{x}_0)\right] \psi(\vec{x}_0) d\vec{x}_0 \left. \right\} \\
 &= \lim_{\{j_0\} \rightarrow \infty} \left\{ \exp\left\{\frac{-itV(\vec{x}_1)}{k}\right\} \left(\frac{1}{4i\pi\epsilon}\right)^{n/2} \right. \\
 &\quad \times \int_{C_{\{j_0\}}} \exp\left[\frac{i\epsilon}{4}\left(\frac{\vec{x}_1 - \vec{x}_0}{\epsilon}\right)^2 + i\bar{\lambda}(\vec{x}_1, \vec{x}_0)\right] \psi(\vec{x}_0) d\vec{x}_0 \left. \right\} \tag{4.8}
 \end{aligned}$$

where $\bar{\lambda}(\vec{x}_1, \vec{x}_0) = \sum_{l=1}^n \bar{\lambda}_l(\vec{x}_1, \vec{x}_0)$, and all limits are taken independently of each other. As mentioned earlier, if the vector potential is well behaved and if \vec{x}_1, \vec{x}_0 are close, we can approximate the last integral in (4.7) and conclude that $\bar{\lambda}(\vec{x}_1, \vec{x}_0)$ is close to $(\vec{x}_1 - \vec{x}_0) * \vec{a}\left(\frac{\vec{x}_1 + \vec{x}_0}{2}\right)$.

Furthermore,

$$\begin{aligned}
 F^k\left(\frac{t}{k}\right)\psi &= \lim_{\{J_{k-1}\} \rightarrow \infty} \left(\frac{1}{4i\pi\epsilon}\right)^{[n(k-1)]/2} \int_{D_{k-1}} \exp\left\{i\epsilon \sum_{j=0}^{k-1} \left[\frac{1}{4}\left(\frac{\vec{x}_{j+1} - \vec{x}_j}{\epsilon}\right)^2 \right. \right. \\
 &\quad \left. \left. - V(\vec{x}_{j+1}) + \frac{\bar{\lambda}(\vec{x}_{j+1}, \vec{x}_j)}{\epsilon}\right]\right\} \psi(\vec{x}_0) d\vec{x}_0 \dots d\vec{x}_{k-1} \\
 &= \left(\frac{1}{4i\pi\epsilon}\right)^{[n(k-1)]/2} \lim_{\{J_{k-1}\} \rightarrow \infty} \int_{D_{k-1}} \exp\{i\epsilon S_k(\epsilon, \vec{x}_0 \dots \vec{x}_k)\} \psi(\vec{x}_0) d\vec{x}_0 \dots d\vec{x}_{k-1} \tag{4.9}
 \end{aligned}$$

where all limits are again taken independently of each other. We see that the vector potential term in (4.9) is not similar to that of (1.1).

Let us use the notation \int_{rK} to denote a Riemann or improper Riemann integral over K and \int_K to denote the Lebesgue integral over K . We now prove our main result in (1.2).

Theorem 4.1. *Let $\phi, \psi \in L^2$ be such that they each have at most a finite number of discontinuities and singularities. Suppose \vec{a} and V satisfies theorem 2.2 or theorem 2.3 and that \vec{a} and V has at most a finite number of discontinuities and singularities. With our previously defined notation, the following holds:*

$$\begin{aligned}
 \langle \phi^*, \exp(-it\bar{H})\psi \rangle_{L^2} &= \int_{\mathbb{R}^n} \phi(\vec{x}) [\exp(-it\bar{H})\psi](\vec{x}) d\vec{x} \\
 &= \lim_{k \rightarrow \infty} \left\{ \left(\frac{1}{4i\pi\epsilon}\right)^{[n(k-1)]/2} \int_{r\mathbb{R}^{n(k+1)}} \phi(\vec{x}_k) \exp\left\{i\epsilon \sum_{j=0}^{k-1} \left[\frac{1}{4}\left(\frac{\vec{x}_{j+1} - \vec{x}_j}{\epsilon}\right)^2 \right. \right. \right. \\
 &\quad \left. \left. - V(\vec{x}_{j+1}) + \frac{\bar{\lambda}(\vec{x}_{j+1}, \vec{x}_j)}{\epsilon}\right]\right\} \psi(\vec{x}_0) d\vec{x}_0 \dots d\vec{x}_k \left. \right\}. \tag{4.10}
 \end{aligned}$$

Proof. Theorem 3.5 and Schwarz’s inequality imply that

$$\left\langle \phi^*, \exp\left(\frac{-it\tilde{H}}{\hbar}\right) \psi \right\rangle_{L^2} = \left\langle \phi^*, \lim_{k \rightarrow \infty} F^k\left(\frac{t}{k}\right) \psi \right\rangle_{L^2} = \lim_{k \rightarrow \infty} \left\langle \phi^*, F^k\left(\frac{t}{k}\right) \psi \right\rangle_{L^2}. \quad (4.11)$$

For any $k \in \mathbb{N}$ with $\epsilon = t/k$, (4.9) implies that

$$\begin{aligned} \left\langle \phi^*, F^k\left(\frac{t}{k}\right) \psi \right\rangle_{L^2} &= \left(\frac{1}{4i\pi\epsilon}\right)^{[n(k-1)]/2} \int_{\mathbb{R}^n} \left[\lim_{\{j_k\} \rightarrow \infty} \chi_{C_{\{j_k\}}} \phi(\vec{x}_k) \right] \\ &\quad \times \left\{ \lim_{\{j_{k-1}\} \rightarrow \infty} \int_{D_{k-1}} \exp\{i\epsilon S_k(\epsilon, \vec{x}_0 \dots \vec{x}_k)\} \psi(\vec{x}_0) d\vec{x}_0 \dots d\vec{x}_{k-1} \right\} d\vec{x}_k \\ &= \left(\frac{1}{4i\pi\epsilon}\right)^{[n(k-1)]/2} \lim_{\{j_k\} \rightarrow \infty} \int_{D_k} \phi(\vec{x}_k) \exp\{i\epsilon S_k(\epsilon, \vec{x}_0 \dots \vec{x}_k)\} \psi(\vec{x}_0) d\vec{x}_0 \dots d\vec{x}_k. \end{aligned} \quad (4.12)$$

In the last equality of (4.12) we used Schwarz’s inequality and took all the L^2 limits outside of the integral as pointwise limits on t . Note that all limits in (4.12) are taken independently of each other. By construction of D_k and the hypothesis on \vec{a} , V , ϕ and ψ , we see that $\phi(\vec{x}_k) \exp\{i\epsilon S_k(\epsilon, \vec{x}_0 \dots \vec{x}_k)\} \psi(\vec{x}_0)$ is a bounded and continuous function on D_k . Thus, we can replace the Lebesgue integral over D_k by a Riemann integral over D_k . By construction of D_k and the fact that all limits are taken independently of each other, we can interpret the limits as improper Riemann integrals. Hence, for all $k \in \mathbb{N}$,

$$\left\langle \phi^*, F^k\left(\frac{t}{k}\right) \psi \right\rangle_{L^2} = \left(\frac{1}{4i\pi\epsilon}\right)^{[n(k-1)]/2} \int_{\mathbb{R}^n} \phi(\vec{x}_k) \exp\{i\epsilon S_k(\epsilon, \vec{x}_0 \dots \vec{x}_k)\} \psi(\vec{x}_0) d\vec{x}_0 \dots d\vec{x}_k. \quad (4.13)$$

The theorem follows from equations (4.13) and (4.11). □

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